Triple systems and associated differences

by

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ABSTRACT

We formulate a problem that is a common generalization of the problems of Skolem and Langford. Necessary conditions on the parameters are derived and many (but not all) cases are solved.

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0. INTRODUCTION

In this paper we study the following problem which is a special case a problem in radioastronomy: how to arrange antennas in a linear array such that certain prescribed mutual distances occur? (see [1] for more details):

PROBLEM I. Let d and m be positive integers. For what values of d and m is it possible to find m triples $A_i = \{a_i, b_i, c_i\}$ $(i = 1,2,\ldots,m)$, such that the 3m numbers (called associated differences of the triples) $b_i - a_i$, $c_i - a_i$, $c_i - b_i$ $(i = 1,2,\ldots,m)$ are all the integers of the set $\{d, d+1, \ldots, d+3m-1\}$?

For example $A_1 = \{0,4,6\}$, $A_2 = \{0,9,10\}$, $A_3 = \{0,8,11\}$, $A_4 = \{0,7,12\}$ is a solution for $m = 4$, $d = 1$.

REMARK. As we are interested only in the differences associated with the triples, we may suppose that $a_i = 0$ in all triples.
Related to this problem is:

PROBLEM II. Let d and m be positive integers. For what values of d and m is it possible to find a partition of the set $\{1,2,\ldots,2m\}$ into m pairs $\{p_i, q_i\}$ such that the m numbers $q_i - p_i$ $(i = 1,\ldots,m)$ are all the integers of the set $\{d, d+1, \ldots, d+m-1\}$?

Obviously a solution to the second problem implies a solution to the first one: take as triples $A_i = (0, p_i+m+d-1, q_i+m+d-1)$.
All our solutions to problem I will also be solutions to problem II.
(But the solution given in the above example is not derived from a solution to problem II.)

PROPOSITION 1. *Necessary conditions for the existence of a solution to problem I are:*

(i) $m \geq 2d-1$ *or* $m = 0$

(ii) *If* d *is odd* $m \equiv 0$ *or* 1 (mod4)
If d *is even* $m \equiv 0$ *or* 3 (mod4).

PROOF. This is a special case of theorem 2.4 of [1]. For completeness we give an independent proof in this case.

Let the triples $\{0,b_i,c_i\}$ $(i = 1,2,\ldots,m)$ constitute a solution to problem I, where $b_i < c_i$. Then

$$\sum_{i=1}^{2m} (d+i-1) \le \sum_{i=1}^{m} b_i+(c_i-b_i) = \sum_{i=1}^{m} c_i \le \sum_{i=1}^{m} (d+3m-i)$$

since all differences b_i, c_i, c_i-b_i have to be different.

Hence $m(2d+2m-1) \le \frac{1}{2}m(2d+5m-1)$ from which (i) follows.

Furthermore

$$\sum_{i=1}^{m} b_i+(c_i-b_i)+c_i = 2\sum_{i=1}^{m} c_i = \sum_{i=1}^{3m} (d+i-1) = \frac{3}{2}\, m(2d+3m-1)$$

is even, so that $3m(2d+3m-1) \equiv 0 \pmod 4$. This yields (ii). □

1. RESULTS

THEOREM 1. *For* d = 1,2, *or* 3 *the necessary conditions of proposition* 1 *are sufficient to guarantee the existence of a solution to problem II (and a fortiori to problem I).*

PROOF. (i) $d = 1$.

In this case problem II reduces to Skolem's problem [7]: for what values of m is it possible to partition the integers $\{1,2,\ldots,2m\}$ into m pairs $\{a_i,b_i\}$ $(i = 1,2\ldots,m)$ such that $b_i-a_i = i$?

But it is well known [4,7] that a solution of Skolem's problem exists iff $m \equiv 0$ or $1 \pmod 4$, and thus case (i) is proved. □

REMARK. Recall that a graceful numbering [3] (or β-valuation [6]) of a graph G with e edges is an assignment of a subset of the numbers $\{0,1,\ldots,e\}$ to the vertices of G in such a way that the values of the edges are all the numbers from 1 to e, where the value of an edge is defined as the absolute value of the difference between the numbers assigned to its endpoints.

Then in case $d = 1$ a solution to problem I is equivalent to a graceful numbering of the graph consisting of m triangles having exactly one vertex in common (this is an easy consequence of the remark in the introduction). The existence of a graceful numbering of such graphs was asked by C. HOEDE (who called these graphs "mills") at the 5th Hungarian Colloquium in Keszthely 1976.

(ii) d = 2

In this case problem II is equivalent to Langford's problem [5]: for what values of m is it possible to find a sequence of length 2m consisting of 2 occurrences of i ($1 \le i \le m$) such that for each i the two occurrences of i are separated by i other elements of the sequence?

EXAMPLE. For m = 3 (3,1,2,1,3,2) is a Langford sequence.

If the number i occurs at positions a_i and b_i in the sequence, then the pairs $\{a_i,b_i\}$ partition $\{1,2,\ldots,2m\}$ while $b_i-a_i = i+1$, i.e. we have a solution of problem II with d = 2. Conversely any solution to problem II with d = 2 yields a Langford sequence. But it has been proved by R.O. DAVIES [2] that a Langford sequence exists iff $m \equiv 0$ or 3 (mod4), and thus case (ii) is proved.

(iii) d = 3

First let m = 4k, k > 1. A solution is given by the following eight groups of pairs $\{a_i,b_i\}$:

(AG1)	a_i	b_i	b_i-a_i	
(1)	j	4k-j+3	4k+2-2j	j = 1,2,...,k
(2)	k+j	3k-j+3	2k+3-2j	j = 2,...,k
(3)	k+1	5k+2	4k+1	
(4)	2k+1	6k+3	4k+2	
(5)	2k+2	6k+1	4k-1	
(6)	4k+2	6k+2	2k	
(7)	4k+j+2	8k-j+1	4k-2j-1	j = 1,...,k-1
(8)	5k+j+2	7k-j+2	2k-2j	j = 1,...,k-2

Next let m = 4k+1. k > 1. A solution is given by:

(AG2)	a_i	b_i	b_i-a_i	
(1)	j	$4k-j+2$	$4k-2j+2$	$j = 1,2,\dots,k$
(2)	$k+1$	$5k+3$	$4k+2$	
(3)	$k+1+j$	$3k-j+2$	$2k-2j+1$	$j = 1,\dots,k-1$
(4)	$2k+1$	$6k+4$	$4k+3$	
(5)	$2k+2$	$6k+3$	$4k+1$	
(6)	$4k+2$	$6k+2$	$2k$	
(7)	$4k+2+j$	$8k-j+3$	$4k-2j+1$	$j = 1,2,\dots,k$
(8)	$5k+3+j$	$7k-j+3$	$2k-2j$	$j = 1,\dots,k-2$

Finally for $m = 5$ a solution is given by $\{1,8\}$, $\{4,10\}$, $\{2,7\}$, $\{5,9\}$, $\{3,6\}$.

REMARK. Another solution for the case $m = 4k+1$ is given in the next theorem. This completes the proof of theorem 1.

THEOREM 2. *Let* $m \equiv 2d-1 \pmod 4$, $m \geq 2d-1$, $d \geq 2$. Then a solution to problem II exists.

PROOF. We distinguish two cases, according to the parity of d. First let d be even, and let $m = 4t+3$.
From $d \geq 2$ and $m \geq 2d-1$ we get $\frac{1}{2}d-1 \geq 0$ and $t-\frac{1}{2}d+1 \geq 0$.
A solution is given by the following ten groups of pairs $\{p_i,q_i\}$:

(AEB1)	q_i	(last value)	p_i	(last value)	q_i-p_i	(last value)	parity	number of pairs
(1)	$2t+d+2+j$	$3t+\frac{1}{2}d+2$	$2t+1-j$	$t+\frac{1}{2}d+1$	$d+1+2j$	$2t+1$	O	$t-\frac{1}{2}d+1$
(2)	$3t+\frac{1}{2}d+3+j$	$4t+3$	$t+\frac{1}{2}d-1-j$	$d-1$	$2t+4+2j$	$4t-d+4$	E	$t-\frac{1}{2}d+1$
(3)	$4t+4+j$	$4t+\frac{1}{2}d+2$	$d-2-j$	$\frac{1}{2}d$	$4t-d+6+2j$	$4t+2$	E	$\frac{1}{2}d-1$
(4)	$4t+\frac{1}{2}d+3$		$2t+\frac{1}{2}d+1$		$2t+2$		E	1
(5)	$4t+\frac{1}{2}d+4+j$	$4t+d+2$	$\frac{1}{2}d-1-j$	1	$4t+5+2j$	$4t+d+1$	O	$\frac{1}{2}d-1$
(6)	$5t+\frac{1}{2}d+4$		$t+\frac{1}{2}d$		$4t+4$		E	1
(7)	$6t+6+j$	$6t+\frac{1}{2}d+5$	$2t+d+1-j$	$2t+\frac{1}{2}d+2$	$4t-d+5+2j$	$4t+3$	O	$\frac{1}{2}d$
(8)	$6t+\frac{1}{2}d+6+j$	$6t+d+4$	$2t+\frac{1}{2}d-j$	$2t+2$	$4t+6+2j$	$4t+d+2$	E	$\frac{1}{2}d-1$
(9)	$6t+d+5+j$	$7t+\frac{1}{2}d+5$	$6t+5-j$	$5t+\frac{1}{2}d+5$	$d+2j$	$2t$	E	$t-\frac{1}{2}d+1$
(10)	$7t+\frac{1}{2}d+6+j$	$8t+6$	$5t+\frac{1}{2}d+3-j$	$4t+d+3$	$2t+3+2j$	$4t-d+3$	O	$t-\frac{1}{2}d+1$
								$4t+3$

Here the variable j ranges from 0 up to and including n-1, where n is the number of pairs.

Next, let d be odd, and let m = 4t+1, d = e-1.

From d ≥ 3 and m ≥ 2d-1 we get $\frac{1}{2}e-2 \geq 0$ and $t-\frac{1}{2}e+1 \geq 0$.

A solution is given by the following ten groups of pairs $\{p_i, q_i\}$:

(AEB2)	q_i	(last value)	p_i	(last value)	q_i-p_i	(last value)	parity	number of pairs
(1)	2t+e+j	3t+½e	2t-j	t+½e	e+2j	2t	E	t-½e+1
(2)	3t+½e+1+j	4t+1	t+½e-2-j	e-2	2t+3+2j	4t-e+3	O	t-½e+1
(3)	4t+2+j	4t+½e	e-3-j	½e-1	4t-e+5+2j	4t+1	O	½e-1
(4)	4t+½e+1		2t+½e		2t+1		O	1
(5)	4t+½e+2+j	4t+e-1	½e-2-j	1	4t+4+2j	4t+e-2	E	½e-2
(6)	5t+½e+1		t+½e-1		4t+2		E	1
(7)	6t+3+j	6t+½e+1	2t+e-1-j	2t+½e+1	4t-e+4+2j	4t	E	½e-1
(8)	6t+½e+2+j	6t+e	2t+½e-1-j	2t+1	4t+3+2j	4t+e-1	O	½e-1
(9)	6t+e+1+j	7t+½e+1	6t+2-j	5t+½e+2	e-1+2j	2t-1	O	t-½e+1
(10)	7t+½e+2+j	8t+2	5t+½e-j	4t+e	2t+2+2j	4t-e+2	E	t-½e+1
								4t+1

In case m ≡ 0 (mod4) we have a solution for large d:

<u>THEOREM 3</u>. *Let* m = 4t, d = 2t-e (e≥0). *Then if* 2d ≥ 3t+1 *a solution to problem II exists.*

<u>PROOF</u>. From 2d ≥ 3t+1 we get t-2e-1 ≥ 0 so that the following seven groups of pairs provide a solution:

(AEB3)	q_i	(last value)	p_i	(last value)	q_i-p_i	(last value)	number of pairs
(1)	8t-j	7t+e+1	2t+e+1+j	3t	6t-e-1-2j	4t+e+1	t-e
(2)	7t+e-j	6t+e+1	3t+2e+2+j	4t+2e+1	4t-e-2-2j	2t-e	t
(3)	6t+e-2j	6t-e	2t-j	2t-e	4t+e-j	4t	e+1
(4)	6t+e-1-2j	6t-e+1	2t+e-j	2t+1	4t-1-j	4t-e	e
(5)	6t-e-1-j	5t+1	1+j	t-e-1	6t-e-2-2j	4t+e+2	t-e-1
(6)	5t-j	4t+2e+2	t+e+1+j	2t-e-1	4t-e-1-2j	2t+3e+3	t-2e-1
(7)	3t+2e+1-j	3t+1	t-e+j	t+e	2t+3e+1-2j	2t-e+1	2e+1
							4t

REMARK. This solution was found using certain linear programming techniques; I do not know whether it can be generalized to $m \equiv 0 \pmod 4$ and arbitrary d (with $m \geq 2d$). In any case the solutions depicted in tables (AEB1) and (AEB2) are much more elegant than the above one. Concerning the LP techniques and the theory of set-addition, these will be the subject of a future paper.

Solutions for small d can be obtained by pasting together other solutions:

PROPOSITION 2. *Suppose we have solutions of problem II with* $(m,d) = (1,d_0+a)$ *and with* $(m,d) = (a,d_0)$. *Then a solution with* $(m,d) = (1+a,d_0)$ *exists.*

PROOF. Let the first solution consist of the pairs $\{p_i,q_i\}$ $(i = 1,\dots,1)$ and the second one of the pairs $\{a_i,b_i\}$ $(i = 1,\dots,a)$. Then the collection of pairs $\{p_i+2a,q_i+2a\}$ $(i = 1,\dots,1)$ together with $\{a_i,b_i\}$ $(i = 1,\dots,a)$ forms a solution of problem II with $(m,d) = (1+a,d_0)$. □

In particular we get:

THEOREM 4. *Let* $m \equiv 0 \pmod 4$, $m \geq 4(2d-1)$. *Then a solution to problem II exists.*

PROOF. Take in the previous proposition $d_0 = d$, $a = 2d-1$, $1 \geq 2(d_0+a)-1$, 1 odd and apply theorems 1 and 2.

Now in order to complete the solution to problem II, we only have to construct a finite number of solutions for any fixed d.
E.g. for d = 4 we have left the cases m = 12,16,20 or 24, and it is easy to provide an explicit solution:

(i) d = 4, m = 12
Take the following pairs:
{5,9}, {19,24}, {4,10}, {6,13}, {15,23}, {12,21}, {8,18}, {11,22}, {2,14}, {7,20}, {3,17}, {1,16}.

(ii) d = 4, m = 16
Take the following pairs:
{27,31}, {25,30}, {4,10}, {8,15}, {6,14}, {23,32}, {7,17}, {11,22}, {12,24}, {16,29}, {5,19}, {13,28}, {2,18}, {9,26}, {3,21}, {1,20}.

(iii) $d = 4$, $m = 20$

Take the following pairs:

{36,40}, {11,16}, {29,35}, {32,39}, {30,38}, {28,37}, {8,18}, {6,17}, {9,21}, {7,20}, {13,27}, {4,19}, {10,26}, {14,31}, {5,23}, {15,34}, {2,22}, {12,33}, {3,25}, {1,24}.

(iv) $d = 4$, $m = 24$

Take the following pairs:

{43,47}, {40,45}, {12,18}, {34,41}, {38,46}, {39,48}, {9,19}, {33,44}, {8,20}, {22,35}, {11,15}, {6,21}, {16,32}, {7,24}, {13,31}, {4,23}, {10,30}, {15,36}, {5,27}, {14,37}, {2,26}, {17,42}, {3,29}, {1,28}.

This proves:

THEOREM 5. *For* $d = 4$ *the necessary conditions of proposition 1 are sufficient to guarantee the existence of a solution to problem II.*

REFERENCES

[1] BERMOND, J.C., A. KOTZIG & J. TURGEON, *On a combinatorial problem of antennas in radioastronomy*, Proc. 5th Hungarian Colloquium Keszthely 1976.

[2] DAVIES, R.O., *On Langford's problem (II)*, Math. Gaz. 43 (1959) 253-255.

[3] GOLOMB, S.N., *How to number a graph*, in *"Graph theory and computing"* (ed. R.C. Read) Acad. Press, New York 1972, pp. 23-37.

[4] HANANI, H., *A note on Steiner triple systems*, Math. Scand. 8 (1960) 154-156.

[5] LANGFORD, C.D., *Problem*, Math. Gaz. 42 (1958) 228.

[6] ROSA, A., *On certain valuations of the vertices of a graph*, in *"Theory of Graphs"* (ed. P. Rosenstiehl). Dunod Gordon Beach 1967, pp. 349-355.

[7] SKOLEM, Th., *On certain distributions of integers in pairs with given differences*, Math. Scand. 5 (1957) 57-68.